Esercitazione

giovedì 31 ottobre ore 16-18

Corso B → Aula A1

Corso A → Aula C

Media e Varianza di v. a.

__Esempio__ - Si lancia un dado equilibrato e sia X il numero uscito

$$X = \begin{cases} 1 & \text{con prob } 1/6 \\ 2 & 1/6 \\ 3 & 1/6 \\ 4 & 1/6 \\ 5 & 1/6 \\ 6 & 1/6 \end{cases}$$

$$\frac{1+2+3+4+5+6}{6} = \frac{21}{6} = \frac{7}{2} = 3.5$$

Stesso esercizio per un dado truccato in modo che la faccia "1" abbia prob. di uscire doppia rispetto alle altre (che hanno tutte la stessa probab.)

$$1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$$

$$\frac{1+1+2+3+4+5+6}{7} = \frac{22}{7}$$

$$X \left\{ \begin{array}{lll} 1 & 2x = 2/7 & 2x + x + x + x + x + x = 7x = 1 \\ 2 & x = 1/7 & x = \frac{1}{7} \\ \vdots & \vdots \quad \vdots & \\ 6 & x = 1/7 & \end{array} \right.$$

$$\underline{1 \cdot \frac{2}{7}} + \underline{2 \cdot \frac{1}{7}} + 3 \cdot \frac{1}{7} + 4 \cdot \frac{1}{7} + 5 \cdot \frac{1}{7} + 6 \cdot \frac{1}{7}$$

$X \qquad x \qquad p_X(x) = P(X = x)$

$$\sum_x{}' x \cdot P(X=x) = \underline{\sum_x{}' x \, p_X(x)}$$

Sia X una v. a. discreta, con

densità $p_X(x) = P(X = x) \quad x \in R$

Definizione Si dice che X ha speranza (matematica) (media, valore medio, atteso, valore atteso) finita se

expectation

$$\sum_x |x| \, p_X(x) < +\infty .$$

In tal caso, si chiama speranza di X il numero

$$E[X] = \sum_x x \, p_X(x)$$

Se $\sum_n |a_n| < \infty$, allora

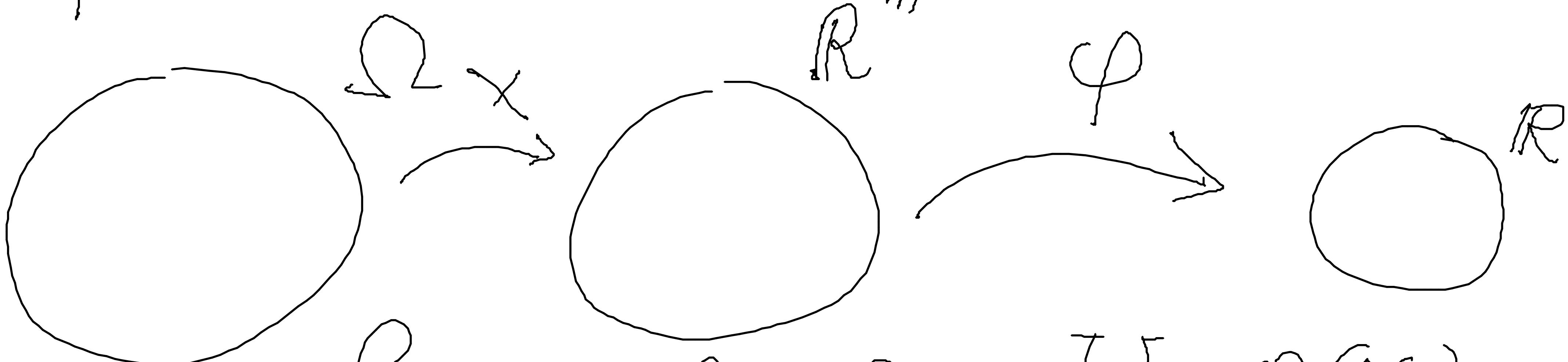
$\sum_n a_n$ converge

$$\sum \left| x \, p_X(x) \right| < \infty \Rightarrow \sum x \, p_X(x)$$

$$\sum |x| \, p_X(x)$$

Teorema Sia $X = (X_1, \dots, X_m)$ un vettore aleatorio m-dimensionale con densità congiunta $p(x) = p(x_1, \dots x_m)$, e sia φ una funzione (**misurabile**)

$$\varphi : \mathbb{R}^m \to \mathbb{R}$$



Cons. la v.a. $U = \varphi(X)$

$$U = \varphi(X) = \varphi(X_1, \dots, X_m).$$

Allora U ha speranza finita se e solo se

$$\sum_x |\varphi(x)|\, p(x) = \underline{\sum_{(x_1,\dots,x_m)} |\underline{\varphi}(x_1,\dots,x_m)|\, \underline{p(x_1,\dots,x_m)}} < \infty$$

e in tal caso

$$E[U] = \sum_x \varphi(x)\, p(x)$$

$$\sum_u \underline{|u|}\, \underline{p_U}(u)$$

1) Quali sono i valori u assunti da U

2) Quanto vale $p_U(u)$ $\forall u$

$$E[U] = \sum_u u \, p_U(u)$$

Proprietà della speranza.

Proposizione. Siano X e Y 2 v.a. aventi entrambe speranza finita. Sia $c \in \mathbb{R}$. Allora

(1) la v.a. $U = cX$ ha speranza finita e $E[U] = c\,E[X]$

(2) la v.a. $Z = X + Y$ ha speranza finita e
$$E[Z] = E[X] + E[Y]$$

Proposizione. Siano X e Y v.a. con speranza finita. Allora

(i) Se $P(X \leq Y) = 1$, allora $E[X] \leq E[Y]$

(ii) $|E[X]| \leq E[|X|]$

esercizio

<u>Proposizione</u>. Siano X e Y 2 v.a. $\boxed{\text{indipendenti}}$, entrambe con speranza finita. Allora la v.a.

$$Z = X \cdot Y \text{ ha speranza finita}$$

e

$$E[Z] = E[X] \cdot E[Y]$$

Dimostraz.

$$Z = \varphi(X, Y) \qquad \varphi(a,b) = a \cdot b$$

$$\sum_{x,y} |\varphi(x,y)|\, p(x,y) \overset{\downarrow}{=}$$

$$= \sum_{x,y} |x \cdot y|\, p(x,y) = \sum_{x,y} |x| \cdot |y|\, p(x,y) =$$

$$= \sum_{x} |x| \sum_{y} |y|\, p(x,y) \overset{\downarrow}{=}$$

$$= \sum_{x} |x| \sum_{y} |y|\, \underbrace{p_X(x)}\, p_Y(y) =$$

$$= \underbrace{\left(\sum_x |x| \, p_X(x) \right)}_{< \infty} \cdot \underbrace{\left(\sum_y |y| \, p_Y(y) \right)}_{< \infty}$$

$$E[XY] = \underline{\underline{E[X] \cdot E[Y]}}$$

Esempi

1) $X \sim B(1, p)$

$$X = \begin{cases} 1 & p \\ 0 & 1-p \end{cases}$$

$$E[X] = \sum_x x \, P(X = x) =$$

$$= 0 \cdot P(X=0) + 1 \cdot P(X=1)$$

$$= 1 \cdot p = p$$

9) $X \sim B(n, p)$

$$P(X=x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0, 1 \ldots, n \\ 0 & \text{———} \end{cases}$$

$$E[X] = \sum_x x\, P(X=x) =$$

$$= \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} = np$$

esercizio

$$X \sim B(n, p)$$

$$X = \sum_{i=1}^{n} X_i$$

$$X_i = \begin{cases} 1 & \text{con prob } p \\ 0 & \text{con prob } (1-p) \end{cases}$$

$$\underline{E[X] = E\left[\sum_{i=1}^{n} X_i\right]}$$

$$E[(X+Y)+Z] = E[(X+Y)] + E[Z]$$

$$= E[X] + E[Y] + E[Z]$$

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] =$$

$$= \sum_{i=1}^{n} \underbrace{E[X_i]}_{=p} = \sum_{i=1}^{n} p = np$$

3) $X \sim \Pi_\lambda \quad \begin{cases} \dfrac{\lambda^x}{x!} e^{-\lambda} & x = 0, 1, 2, \dots \\ 0 & \text{——} \end{cases}$

$$\sum_x |x| \, p_X(x) < \infty$$

$$= \sum_{x=0}^{\infty} x \frac{\lambda^x}{x!} e^{-\lambda} = \sum_{x=1}^{\infty} \cancel{x} \frac{\lambda^x}{\cancel{x}!} e^{-\lambda}$$

$$= \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} e^{-\lambda} =$$

$$= \lambda \underbrace{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda}}_{=1} = \lambda$$

$$\underbrace{\sum_{k=2}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} = \underbrace{\sum_{k=0}^{\infty} \cdots}\ - e^{-\lambda} - \lambda e^{-\lambda}}$$

4) $X \sim$ geometrica di par. p

$$P(X=x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, \dots \\ 0 & \end{cases}$$

$$\sum_{x=1}^{\infty} x\, p\, (1-p)^{x-1} =$$

$$\frac{d}{dy} y^{x} = x\, y^{x-1}$$

$$= p \sum_{x=1}^{\infty} \boxed{\frac{d}{dy}} y^{x} \Big|_{y=1-p} = p \frac{d}{dy} \left(\sum_{x=1}^{\infty} y^{x} \right) \Big|_{y=1-p}$$

attenzione!

$$= p \frac{d}{dy}\left(\frac{y}{1-y}\right) =$$

$$= p \frac{(1-y)+y}{(1-y)^2} = p \left.\frac{1}{(1-y)^2}\right|_{y=1-p}$$

$$= \frac{p}{p^2} = \frac{1}{p} \qquad \boxed{E[X] = \frac{1}{p}}$$

5) X ipergeom. di par.
a, b, n

a oggetti di tipo 1

b oggetti di tipo 2

n estrazioni senza r.

$X = $ n° di oggetti di tipo 1
estratti

$$X_i = \begin{cases} 1 & \text{se esce un oggetto di tipo 1 all'iesima estrazione} \\ 0 & \text{———} \end{cases}$$

$$X = \sum_{i=1}^{n} X_i$$

$$E[X] = \sum_{i=1}^{n} E[X_i] = \frac{na}{a+b}$$

$$E[X_i] = \frac{a}{a+b} \quad \forall i$$

$$E[X] = \sum_{x=0}^{i} x \frac{\binom{a}{x} \cdot \binom{b}{n-x}}{\binom{a+b}{n}}$$

$$= \frac{n\,a}{a+b}$$

$i = 2$

$$E[X_2] = 1 \cdot P(X_2=1) = P(X_2=1)$$

$$= \frac{a}{a+b}$$

$$P(X_2 = 1) = \frac{a}{a+b}$$

$$P(\underbrace{X_2=1}, \underbrace{X_1=1}) + P(\underbrace{X_2=1}, \underbrace{X_1=0})$$

$$= P(X_2=1 \mid X_1=1) \underbrace{P(X_1=1)} +$$

$$+ P(X_2=1 \mid X_1=0) \underbrace{P(X_1=0)}$$

$$= \underbrace{P(X_2 = 1 \mid X_1 = 1)}_{\frac{a-1}{a+b-1}} \underbrace{P(X_1 = 1)}_{\frac{a}{a+b}} +$$

$$+ \underbrace{P(X_2 = 1 \mid X_1 = 0)}_{} \cdot P(X_1 = 0)$$

$$\frac{a}{a+b-1} \cdot \frac{b}{a+b} =$$

$$= \frac{a-1}{a+b-1} \cdot \frac{a}{a+b} + \frac{a}{a+b-1} \cdot \frac{b}{a+b} =$$

$$= \frac{a\,(a-1+b)}{(a+b)\cancel{(a+b-1)}}$$

$$= \frac{a}{a+b}$$

$$= p \frac{d}{dy} \left(\sum_{x=1}^{\infty} y^{x} \right) =$$

$$= p \frac{d}{dy} \left(y \sum_{x=1}^{\infty} y^{x-1} \right) =$$

$$= p \frac{d}{dy} \left(y \underbrace{\sum_{h=0}^{\infty} y^{h}} \right) =$$

$$= p \frac{d}{dy} \left(y \cdot \frac{1}{1-y} \right) =$$

Dimostrazione $\quad p(x,y) =$ dens. cong. di (X,Y) $= P(X=x, Y=y)$

(2) $\quad Z = X+Y = \varphi(X,Y) \qquad \varphi(a,b) = a+b$

$$\sum_{x,y} |\varphi(x,y)|\, p(x,y) = \sum_{x,y} |x+y|\, p(x,y) \leq$$

$$\leq \sum_{x,y} \left(|x| + |y| \right) p(x,y) =$$

$$= \sum_{x,y} |x|\, p(x,y) + \sum_{x,y} |y|\, p(x,y) =$$

$$E[Z] = \sum_{x,y} \varphi(x,y)\, p(x,y) =$$

$$= \sum_{x,y} (x+y)\, p(x,y) =$$

$$= \sum_{x,y} x\, p(x,y) + \sum_{x,y} y\, p(x,y) =$$

$$= \sum_{x} x \underbrace{\sum_{y} p(x,y)}_{} + \sum_{y} y \underbrace{\sum_{x} p(x,y)}_{} =$$

$$= \sum_x{}' \; x \, p_X(x) + \sum_y{}' \; y \, p_Y(y) =$$

$$= E[X] + E[Y]$$

$$E[X+Y] = E[X] + E[Y]$$

(1) $\quad U = cX \qquad U = \varphi(X) \qquad \varphi(a) = ca$

contin per es.

$$\sum_x{}' \sum_y{}' |x|\, p(x,y) + \sum_y{}' \sum_x{}' |y|\, p(x,y) =$$

$$= \sum_x{}' |x| \underbrace{\sum_y{}' p(x,y)}_{= p_X(x)} + \sum_y{}' |y| \underbrace{\sum_x{}' p(x,y)}_{= p_Y(y)} =$$

$$= \underbrace{\sum_x{}' |x|\, p_X(x)}_{< \infty} + \underbrace{\sum_y{}' |y|\, p_Y(y)}_{< \infty}$$